Feedback Linearization

ECEN 5638 - Controls Lab

Riley Kenyon

04/30/2020

Theory

Feedback linearization can be performed by manipulating a system into controller form, shown by

$$\dot{x} = \mathrm{Ax} + B\left(\psi(x) + \Gamma(x)u\right),$$

Where $\Gamma(x)$ is an invertible matrix, $\psi(x)$ is a vector of the same dimension as u, and (A,B) are apprpriately sized matrices that satisfy the controllability conditions. By choosing

$$u = \Gamma(x)^{-1}(v - \psi(x)),$$

we obtain the linear system

$$\dot{x} = \mathrm{Ax} + \mathrm{Bv}$$

Which we can now design a controler around as a linear system.

Simple Pendulum

Consider the simple mendulum model we controlled in the lab

$$\mathrm{ml}^2\ddot{\alpha} = -\mathrm{mgl}\sin(\alpha) - c\dot{\alpha} + \mathrm{ku}$$

1. Write the system in controller form by identifying A,B, $\psi(x)$, $\Gamma(x)$

The equation can be re-written as the following

$$\ddot{\alpha} = -\frac{c}{\mathrm{ml}^2}\dot{\alpha} + \frac{k}{\mathrm{ml}^2}\left(-\frac{\mathrm{mgl}}{k}\sin(\alpha) + u\right)$$

and in matrix form as

$$\begin{bmatrix}\dot{\alpha}\\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -\frac{c}{\mathrm{ml}^2}\end{bmatrix}\begin{bmatrix}\alpha\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}0\\ \frac{k}{\mathrm{ml}^2}\end{bmatrix}\left(\left[-\frac{\mathrm{mgl}}{K}\sin(\alpha)\right] + [1]u\right)$$

where the identified matrices are represented by

$$A = \begin{bmatrix}0 & 1\\ 0 & -\frac{c}{\mathrm{ml}^2}\end{bmatrix}, \quad B = \begin{bmatrix}0\\ \frac{k}{\mathrm{ml}^2}\end{bmatrix}, \quad \psi(\alpha) = -\frac{\mathrm{mgl}}{k}\sin(\alpha), \quad \Gamma(x) = 1\,,$$

2. Prove that the PD with Gravity Compensation law is a form of feedback linearization

Recall the PD with gravity compensation is a law represented by

$$u = \frac{\text{mgl}}{k}\sin(\alpha) - K_p(\theta - \theta_r) - K_D\dot{\theta}$$

Using feedback linearization and choosing u as described above, the PD with gravity compensation is equal to

$$u = 1\left(\frac{\text{mgl}}{K}\sin(\alpha) + v\right),$$

where

$$v = -K_p(\alpha - \alpha_r) - K_D\dot{\alpha}$$

Mixing Tank

Consider a mixing tank fed by two inlets. Let u_1 be the incoming flow of a solvent (water) and let u_2 be the incoming flow of solute (syrup). Let x_1 be the total volume of liquid in the tank and let x_2 be the total concentration of the solution. Assuming a natural outlet flow, the total volume in the tank is governed by

$$\dot{x}_1 = u_1 + u_2 - \kappa\sqrt{x_1}$$

The solvent concentration is instead equal to

$$x_2 = \frac{\int u_2 - x_2\kappa\sqrt{x_1}}{x_1}.$$

1. Compute the state-space model of the system

Taking the derivative with respect to time results in the following for the second equation

$$\dot{x}_1 x_2 + x_1\dot{x}_2 = u_2 - x_2\kappa\sqrt{x_1}$$

written in terms of $\dot{x}_2$,

$$\dot{x}_2 = -\frac{2\kappa x_2}{\sqrt{x_1}} + \frac{(1 - x_2)}{x_1}u_2 - \frac{x_2}{x_1}u_1$$

Separated and written in state-space form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}-\kappa\sqrt{x_1}\\ -\dfrac{2\kappa x_2}{\sqrt{x_1}}\end{bmatrix} + \begin{bmatrix}1 & 1\\ -\dfrac{x_2}{x_1} & \dfrac{1 - x_2}{x_1}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}\right)$$

2. Written in controller form the matrices correspond to the following

$$A = \begin{bmatrix}0\\ 0\end{bmatrix},\quad B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \psi(x) = \begin{bmatrix}-\kappa\sqrt{x_1}\\ -\dfrac{2\kappa x_2}{\sqrt{x_1}}\end{bmatrix},\ \Gamma(x) = \begin{bmatrix}1 & 1\\ -\dfrac{x_2}{x_1} & \dfrac{1 - x_2}{x_1}\end{bmatrix}.$$

3. Using feedback linearization, design a controller that stabilizes the system to the refereces $x_1 = r_1 > 0$. and $x_2 = r_2 \in [0, 1]$.

```
% Non linear controller for tank
k = 10;          % Flow rate
A = zeros(2);
B = eye(2);

% Simulation
T = 10;
IC = [1 0]';
REF = [5 0.3]';     % Note that higher concentrations are not achievable due to the lower limit

%% Design controller LQR
R11 = 0.1;                          % affets u(1)
R22 = 0.1;                         % affects u(2)
R = diag([1/R11,1/R22]);

Q11 = 0.01;                        % affects x(1)
Q22 = 0.001;                       % affects x(2)
Q = diag([1/Q11,1/Q22]);

[K,~,~] = lqr(A,B,Q,R);
disp(K)
```

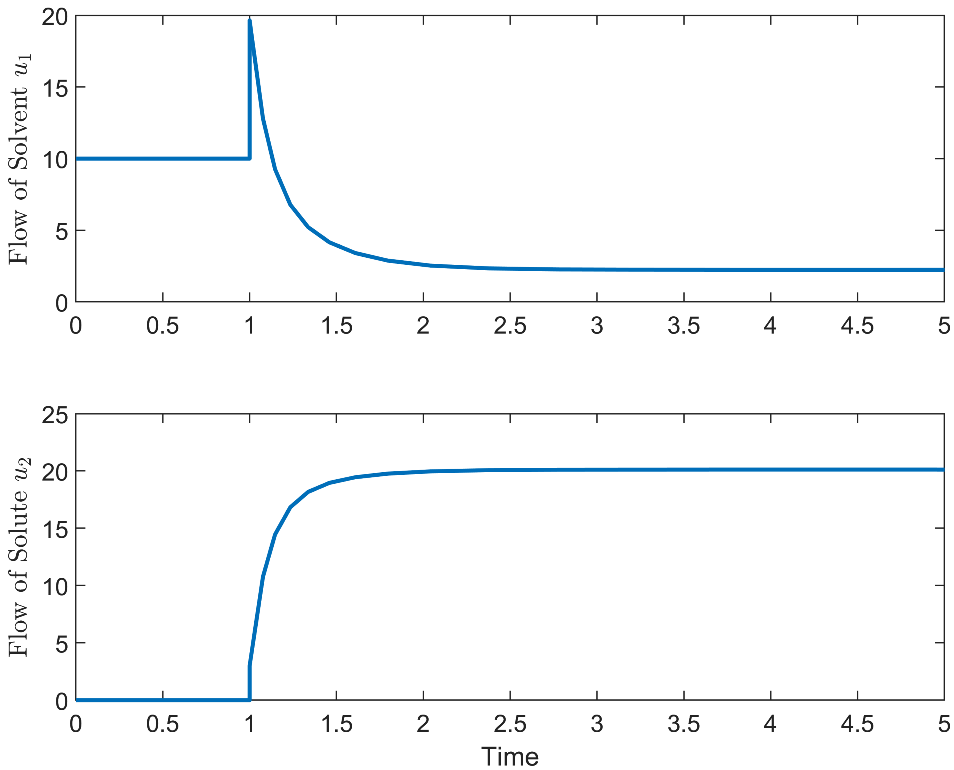
```
    3.1623   -0.0000
    0.0000   10.0000
```

4. Simulate the closed-loop dynamics

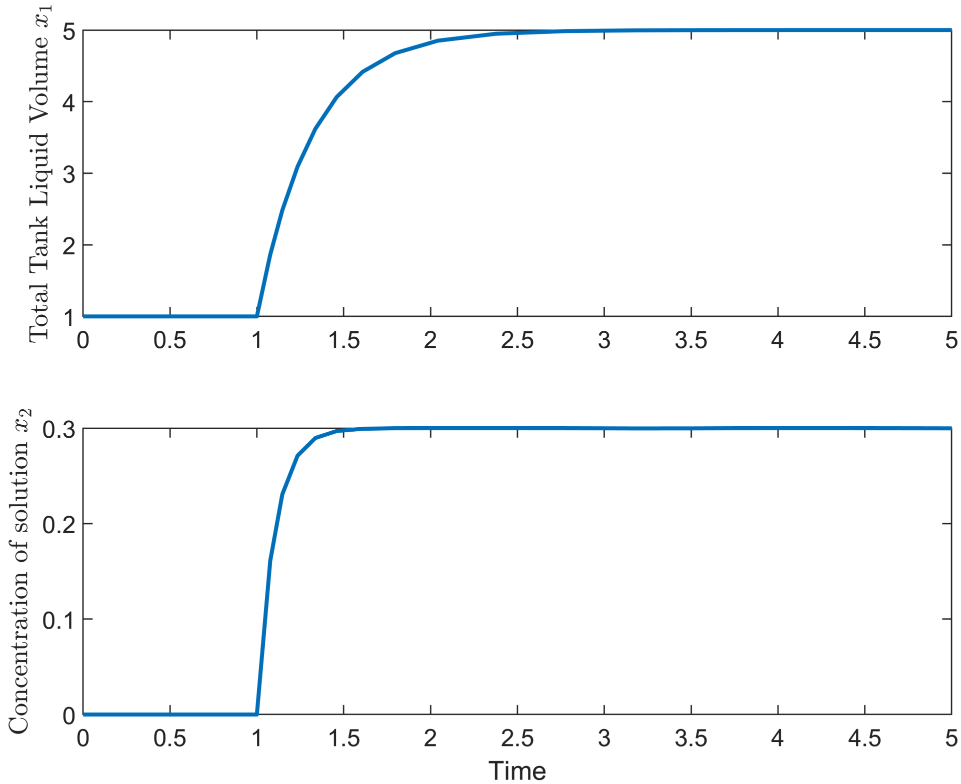
```
data = sim('tank_sim');
```

```
Warning: Convergence problem (mode oscillation) detected when solving algebraic loop containing 'tank_sim/
Plant/Plant' at time 0.4.  Simulink will try to solve this loop using Simulink 3 (R11) strategy.  Use
 feature('ModeIterationsInAlgLoops',0) to disable the strategy introduced in Simulink 4 (R12)
```

```
figure(1)
subplot(2,1,1)
plot(data.t,data.u(:,1),'LineWidth',1.5)
xlim([0 5])
ylabel('Flow of Solvent $u_1$',"Interpreter","latex")
subplot(2,1,2)
plot(data.t,data.u(:,2),'LineWidth',1.5)
xlim([0 5])
ylabel('Flow of Solute $u_2$',"Interpreter","latex")
xlabel('Time')
```




```
figure(2)
subplot(2,1,1)
plot(data.t,data.x(:,1),'LineWidth',1.5)
xlim([0 5])
ylabel('Total Tank Liquid Volume $x_1$',"Interpreter","latex")
subplot(2,1,2)
plot(data.t,data.x(:,2),'LineWidth',1.5)
xlim([0 5])
ylabel('Concentration of solution $x_2$',"Interpreter","latex")
xlabel('Time')
```


Total Tank Liquid Volume x_1
1
2
3
4
5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Concentration of solution x_2
0
0.1
0.2
0.3
Time